

Structural VAR Models and Applications

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SVAR: Objectives

- Whereas the VAR model is able to capture efficiently the interactions between the different variables, it does not allow to reveal the underlying causal mechanisms since two different causal schemes can correspond to the same reduced forms.
- By taking into account certain economic relationships, a Structural VAR model (SVAR) makes it possible to identify structural shocks while letting play the interactions between the different variables

Overview of the presentation

- ① Structural Vector Auto-Regressions definitions
 - Motivations and Definitions
 - Causal mechanisms
- ② Identification issues
 - Cholesky
 - Others
- ③ Applications
 - ① Dedola and Lippi
 - ② Kilian (AER, 2009)

Motivations

Assume 2 stationary processes (x_t) and (z_t) .

We treat each variable symmetrically within the contemporaneous bivariate system:

$$\begin{cases} x_t = b_{10} - b_{12}z_t + \gamma_{11}x_{t-1} + \gamma_{12}z_{t-1} + u_{xt} \\ z_t = b_{20} - b_{21}x_t + \gamma_{21}x_{t-1} + \gamma_{22}z_{t-1} + u_{zt} \end{cases} \quad (1)$$

where u_{xt} and u_{zt} are **pure innovation shocks**, no cross-correlated at all leads and lags k , ie:

$$u_{xt} \sim WN(\sigma_x^2)$$

$$u_{zt} \sim WN(\sigma_z^2)$$

$$\rho(u_{xt}, u_{zt}) = 0$$

$$\rho(u_{xt}, u_{z,t-k}) = 0$$

Motivations

Definition:

This equation is a 1st order VAR process in its *primitive form*:

$$\begin{cases} x_t = b_{10} - b_{12}z_t + \gamma_{11}x_{t-1} + \gamma_{12}z_{t-1} + u_{xt} \\ z_t = b_{20} - b_{21}x_t + \gamma_{21}x_{t-1} + \gamma_{22}z_{t-1} + u_{zt} \end{cases}$$

Role of coefficients:

- $-b_{12}$ = contemporaneous effect of unit change in z_t on x_t
- γ_{12} = lagged effect of unit change in z_{t-1} on x_t
- if $b_{21} \neq 0$ then a shock u_{xt} has an indirect contemporaneous effect on z_t

Motivations

This system is **not** a *reduced form* standard VAR equation as z_t has a contemporaneous effect on x_t and conversely.

Let's rewrite equation (1):

$$\begin{cases} x_t + b_{12}z_t = b_{10} + \gamma_{11}x_{t-1} + \gamma_{12}z_{t-1} + u_{xt} \\ b_{21}x_t + z_t = b_{20} + \gamma_{21}x_{t-1} + \gamma_{22}z_{t-1} + u_{zt} \end{cases} \quad (2)$$

Motivations

Then in a matricial form:

$$B^{-1}y_t = \Gamma_0 + \Gamma_1 y_{t-1} + u_t, \quad (3)$$

with : $y'_t = (x_t, z_t)$, $\Gamma'_0 = (b_{10}, b_{20})$ and $u'_t = (u_{xt}, u_{zt})$

and :

$$B^{-1} = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}$$

$$\Gamma_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

Recall

If we assume M is a 2×2 matrix such as :

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $ad - bc \neq 0$ then M can be inverted and

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Motivations

Assuming B^{-1} is invertible, by multiplying by B the *primitive form* can be rewritten in its *reduced form*:

$$y_t = A_0 + A_1 y_{t-1} + \varepsilon_t \quad (4)$$

with

$$A_0 = B\Gamma_0,$$

$$A_1 = B\Gamma_1,$$

$$\varepsilon_t = Bu_t$$

Motivations

Thus the reduced form can be rewritten as :

$$\begin{cases} x_t = a_{10} + a_{11}x_{t-1} + a_{12}z_{t-1} + \varepsilon_{xt} \\ z_t = a_{20} + a_{21}x_{t-1} + a_{22}z_{t-1} + \varepsilon_{zt} \end{cases} \quad (5)$$

where residuals ε_t are linear combinations of the 2 structural shocks such as:

$$\begin{cases} \varepsilon_{xt} = \frac{1}{1 - b_{21}b_{12}}(u_{xt} - b_{12}u_{zt}) \\ \varepsilon_{zt} = \frac{1}{1 - b_{21}b_{12}}(u_{zt} - b_{12}u_{xt}) \end{cases} \quad (6)$$

Properties of ε_t

We can show that

1

$$E(\varepsilon_{xt}) = E(\varepsilon_{zt}) = 0$$

2

$$E(\varepsilon_{xt}^2) = \frac{1}{(1 - b_{12}b_{21})^2}(\sigma_x^2 + b_{12}^2\sigma_z^2)$$

$$E(\varepsilon_{zt}^2) = \frac{1}{(1 - b_{12}b_{21})^2}(\sigma_z^2 + b_{21}^2\sigma_x^2)$$

3 For all $j \neq 0$:

$$E(\varepsilon_{xt}\varepsilon_{x,t-j}) = 0$$

$$E(\varepsilon_{zt}\varepsilon_{z,t-j}) = 0$$

Properties of ε_t

What about covariance? We can show that:

$$E(\varepsilon_{xt}\varepsilon_{zt}) = -\frac{1}{(1 - b_{12}b_{21})^2}(b_{21}\sigma_x^2 + b_{12}\sigma_z^2)$$

In general, non-null covariance except if $b_{12} = b_{21} = 0$, that is no contemporaneous effect.

Covariance matrix Ω such that:

$$\Omega = \begin{pmatrix} E(\varepsilon_{xt}^2) & E(\varepsilon_{xt}\varepsilon_{zt}) \\ E(\varepsilon_{xt}\varepsilon_{zt}) & E(\varepsilon_{zt}^2) \end{pmatrix}$$

Identification issue

The residuals ε_t have *good* properties, meaning that the reduced form can be efficiently estimated.

Question: Is it possible to recover all the information present in the *primitive form* from the estimation of *the reduced form* ?

Answer: NO ! 9 parameters in the reduced form but 10 parameters in the primitive form : **The system is not fully identified**

Identification issue

Solution (Sims, 1980): recursive identification scheme, ie :
recursive system with restrictions on the primitive form.

Ex. Assume z_t does not have any contemporaneous effect on x_t ,
ie : $b_{21} = 0$

Thus B^{-1} is a lower diagonal matrix:

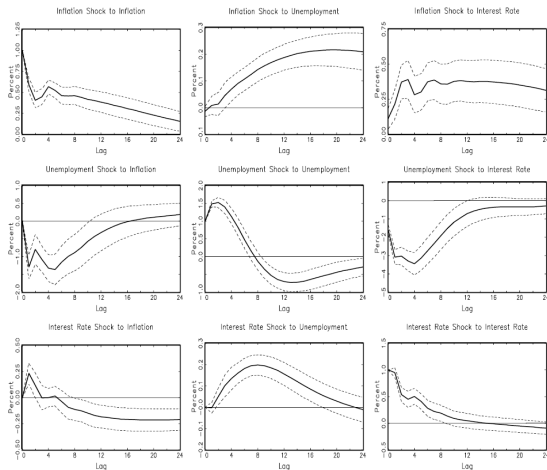
$$B^{-1} = \begin{pmatrix} 1 & 0 \\ b_{21} & 1 \end{pmatrix}$$

Alternative: $b_{12} = 0$, ie: x_t has no effect on z_t

Main issue in SVAR modelling: how to define B^{-1} ?

Causal mechanisms in a stylized economy

IRFs from SVAR with Inflation, Unemployment and FFR



Causal mechanisms in a stylized economy

- Assume that the GDP growth g_t is affected by some real shocks $u_{r,t}$ following:

$$g_t = -0.3(i_{t-1} - \pi_t) + u_{r,t}$$

where i_t is the nominal interest rate, π_t is the inflation rate.

- Besides, assume that we have a simple Taylor rule governing short-term interest rates :

$$i_t = 0.9i_{t-1} + 1.5\pi_t + u_{mp,t}$$

where $u_{mp,t}$ is a monetary-policy shock.

- The inflation rate is supposed to follow a backward-looking Phillips curve:

$$\pi_t = 0.9\pi_{t-1} + 0.2g_{t-1} + u_{n,t}$$

where $u_{n,t}$ is a cost-push shock.

Causal mechanisms in a stylized economy

- The structural shocks $u_t = (u_{r,t}, u_{mp,t}, u_{n,t})$ are uncorrelated (i.e., the covariance matrix of u_t , denoted with Ω_u is diagonal)
- The structural shocks u_t are serially uncorrelated (i.e. $\text{Cov}(u_{t-k}, u_t) = 0$ for any t and $k > 0$).

Causal mechanisms in a stylized economy

- The “structural” model reads

$$\begin{cases} g_t &= -0.3(i_{t-1} - \pi_t) + u_{r,t} \\ i_t &= 0.9i_{t-1} + 1.5\pi_t + u_{mp,t} \\ \pi_t &= 0.9\pi_{t-1} + 0.2g_{t-1} + u_{n,t}. \end{cases}$$

- To get it in a “reduced-form”, let us substitute π_t in the right-hand sides of the first two equations:

$$\begin{cases} g_t &= 0.06g_{t-1} - 0.3i_{t-1} + 0.27\pi_{t-1} + 0.3u_{n,t} + u_{r,t} \\ i_t &= 0.9i_{t-1} + 1.35\pi_{t-1} + 0.3g_{t-1} + u_{mp,t} + 1.5u_{n,t} \\ \pi_t &= 0.9\pi_{t-1} + 0.2g_{t-1} + u_{n,t}. \end{cases}$$

Causal mechanisms in a stylized economy

- In matrix form

$$\begin{bmatrix} g_t \\ i_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 0.06 & -0.3 & 0.27 \\ 0.3 & 0.9 & 1.35 \\ 0.2 & 0 & 0.9 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ i_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{g,t} \\ \varepsilon_{i,t} \\ \varepsilon_{\pi,t} \end{bmatrix}$$

$$\text{with } \begin{bmatrix} \varepsilon_{g,t} \\ \varepsilon_{i,t} \\ \varepsilon_{\pi,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.3 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{r,t} \\ u_{mp,t} \\ u_{n,t} \end{bmatrix} = B \begin{bmatrix} u_{r,t} \\ u_{mp,t} \\ u_{n,t} \end{bmatrix}.$$

- With the procedure described above, one only gets an estimate of Ω_ε where

$$\Omega_\varepsilon = \text{Var} \left(\begin{bmatrix} \varepsilon_{g,t} \\ \varepsilon_{i,t} \\ \varepsilon_{\pi,t} \end{bmatrix} \right).$$

Identification in SVAR

We saw that a structural model can be thought in its general form :

$$\varepsilon_t = y_t - E(y_t | I_{t-1}), \quad (7)$$

with

$$\varepsilon_t = Bu_t \quad (8)$$

In our case of $SVAR(p)$, equation (7) is:

$$y_t = A_0 + \sum_{i=1}^p A_i y_{t-i} + \varepsilon_t, \quad (9)$$

or

$$A(L)y_t = \varepsilon_t, \quad (10)$$

SVAR: Orthogonalization

- Estimation of AR parameters via MLE then estimation of structural parameters in B
- Note that we must have

$$\Omega_\varepsilon = B\Omega_u B'$$

where Ω_u is diagonal positive such as: $\Omega_u = I$.

- In addition, given the “structural” framework, one knows that B is a triangular matrix (lower or upper according to the specification).

SVAR: Orthogonalization

- It has been shown previously that a SVAR is a structural model that draws from a theoretical framework.
- As a starting point, we always have $\Omega_\varepsilon = B\Omega_u B'$ that provides us with $n(n+1)/2$ restrictions to recover the B matrix.
- Consequently, to get the B matrix, one have to impose $n(n-1)/2$ additional restrictions.

SVAR: Restrictions

There exist several restrictions that can be implemented in a SVAR:

- a **short-run** restriction prevents a structural shock from affecting an endogenous variable contemporaneously by setting to zero some entries of B .
- a **long-run** restriction prevents a structural shock from affecting an endogenous variable in a cumulative way.
- a **sign** restriction imposes negative or positive parameters in matrix B .

SVAR: Short-Run restrictions

- Short-run restrictions are simpler to implement than the other ones.
- There are particular cases in which some well-known matrix decomposition can be used to easily estimate some specific SVAR.
- Imagine a context in which you can argue that there exists an “ordering” of the shocks:
 - A first shock structural $u_{1,t}$ can affect instantaneously (i.e., in t) only one of the endogenous variable (say, $y_{1,t}$) through $\varepsilon_{1,t}$;
 - A second shock, $u_{2,t}$ can affect instantaneously the first two endogenous variables (say, $y_{1,t}$ and $y_{2,t}$) through $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$;
 - ...

SVAR: Short-Run restrictions

Within this framework, the structural recursive mechanism of variable ordering allows the creation of the lower-triangular B matrix:

$$\begin{cases} \varepsilon_{1,t} &= u_{1,t} \\ \varepsilon_{2,t} &= \beta_{21}u_{1,t} + u_{2,t} \\ \varepsilon_{3,t} &= \beta_{31}u_{1,t} + \beta_{32}u_{2,t} + u_{3,t} \\ \dots &= \dots \\ \varepsilon_{n,t} &= \beta_{n1}u_{1,t} + \dots + \beta_{n,n-1}u_{n-1,t} + u_{n,t} \end{cases}$$

where the u_t are the **structural shocks** and are not cross-correlated at all leads and lags.

SVAR: Short-Run restrictions

The triangular structural model imposes the recursive causal ordering:

$$y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_n$$

- y_1 causes $y_2, y_3, \dots, y_n$
- but $y_2, y_3, \dots, y_n$ do not cause y_1
- y_2 causes $y_3, \dots, y_n$
- but $y_3, \dots, y_n$ do not cause y_2
- ...

Obviously, a different ordering of variables leads to a different result

Cholesky decomposition/factorization

Decomposition of a symmetric positive definite matrix M into a lower-triangular matrix and its conjugate transpose (by André-Louis Cholesky), ie:

$$M = LL'$$

with

$$L = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}$$

Cholesky decomposition/factorization

Property:

The ML estimate of B is the Cholesky decomposition of $\hat{\Omega}_\varepsilon$ the sample covariance matrix of VAR residuals.

If the model is just-identified, $\hat{\Omega}_\varepsilon(BB')^{-1} = I_n$ and the log-likelihood simplifies to:

$$L = \text{Const} - 0.5 \log |\hat{\Omega}_\varepsilon| - 0.5n$$

IRF for SVAR

If $A(L)$ is invertible in eq. (10) then the VMA representation is:

$$y_t = A(L)^{-1}\varepsilon_t = \Theta(L) = \varepsilon_t + \Theta_1\varepsilon_{t-1} + \dots \quad (11)$$

thus the *structural* VMA representation is:

$$y_t = Bu_t + \Theta_1Bu_{t-1} + \dots = Q_0u_t + Q_1u_{t-1}\dots \quad (12)$$

Thus, the IRF:

$$\mathcal{I}_{i,j,h} = \frac{\partial y_{i,t+h}}{\partial u_{j,t}} \quad (13)$$

$$= (Q_h)_{ij} \quad (14)$$

Confidence intervals are generally computed by Bootstrap

Forecast Error Variance Decomposition

FEV after h steps:

$$\Omega_h = \sum_{k=0}^h Q_k Q_k' \quad (15)$$

Hence the variance v_i for y_i is :

$$v_i = (\Omega_h)_{ii} = \sum_{k=0}^h e_i' Q_k Q_k' e_i = \sum_{k=0}^h \sum_{l=1}^n (q_{i,l}^k)^2$$

where e_i is a zero-vector with 1 at the i^{th} element and $(q_{i,l}^k)^2$ is the (i, l) element of Q_k .

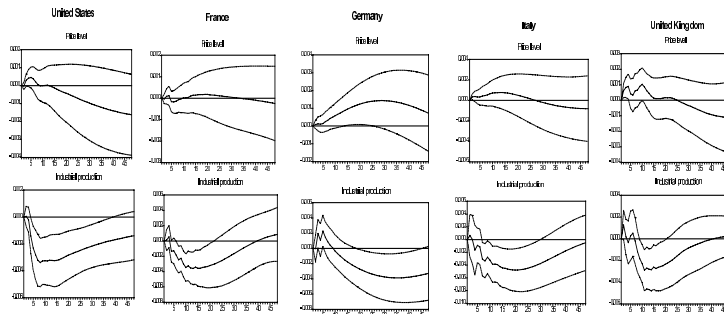
Thus the share of variance on y_i attributed to the j^{th} shock after h periods is:

$$\mathcal{VD}_{i,j,h} = \frac{\sum_{k=0}^h (q_{i,l}^k)^2}{\sum_{k=0}^h \sum_{l=1}^n (q_{i,l}^k)^2}$$

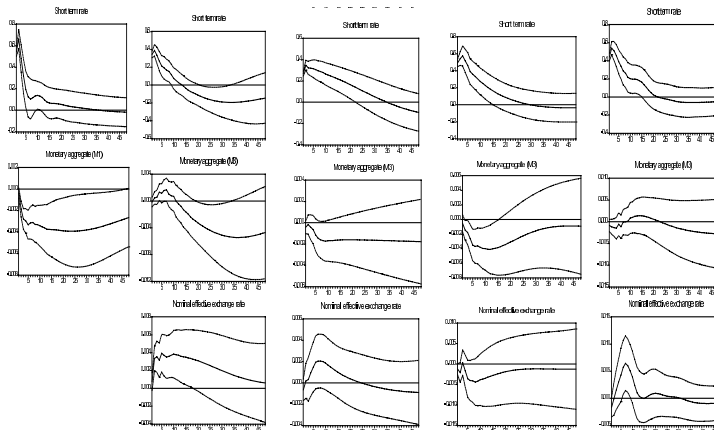
Cholesky decomposition: Illustration

- Dedola and Lippi ("*The Monetary Transmission Mechanism: Evidence from the Industries of Five OECD Countries*", 2005) estimate 5 SVAR for the US, the UK, Germany, France and Italy to analyse the monetary-policy transmission mechanisms
- SVAR($p = 5$) estimation over the period 1975-1997
- Identification scheme is based on Cholesky decompositions, the ordering of the endogenous variables being: the industrial production, the consumer price index, a commodity price index, the short-term rate and a monetary aggregate (except for the US).
- This ordering implies that monetary policy reacts to the shocks affecting the first three variables but that the latter react to monetary policy with a one-period lag.

Responses of Macro-variables to a monetary policy shock



Responses of Macro-variables to a monetary policy shock



Kilian (2009)

- *"Not all shocks are alike: Disentangling demand from supply shocks in the crude oil market"* AER.
- **Objective:** Distinct effects of supply and demand shocks on oil prices
- **Methodology:** A structural monthly VAR is used to identify aggregate supply of oil, aggregate demand of oil and precautionary shocks
- **Findings:** Precautionary shocks have adverse effects on oil prices .

Kilian (2009)

Data

- Global oil production
- Global demand estimated by monthly industrial commodities
- Real price of oil

Model:

$$A_0 y_t = \alpha + \sum_{i=1}^{24} A_i y_{t-i} + \varepsilon_t$$

where

$$y_t = (\Delta Prod_t, rea_t, rpo_t)'$$

Kilian (2009)

$$\begin{bmatrix} u_{Prod,t} \\ u_{Rea,t} \\ u_{Rpo,t} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{Oilsupply,t} \\ \varepsilon_{Aggregateddemand,t} \\ \varepsilon_{Oilspecific,t} \end{bmatrix}$$

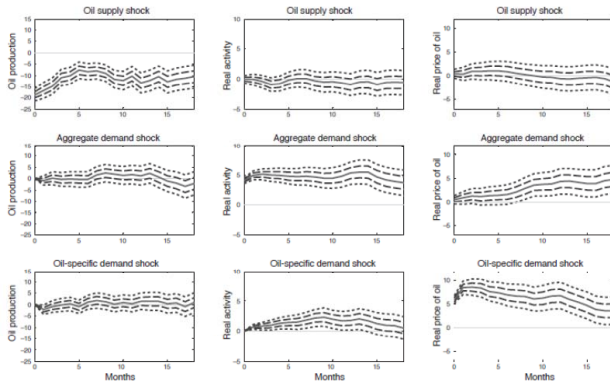


FIGURE 3. RESPONSES TO ONE-STANDARD-DEVIATION STRUCTURAL SHOCKS

(Point estimates with one- and two-standard error bands)

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